Joint probability distributions and multipoint correlations of the continuous-time random walk

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We present an efficient method to determine the Fourier-Laplace transform of the joint n-point probability distribution of a continuous-time random walk for arbitrary finite n. Additionally, we devise a recursive procedure with which it is possible to calculate the Laplace transforms of the multipoint correlation functions without having to determine the joint probability distributions first. The methods are used on several examples with both independent and dependent distributions for the waiting time and the spatial step size.

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I. INTRODUCTION

Since the introduction of the continuous-time random walk (CTRW) by Montroll and Weiss [1], its concept has been successfully applied to model subdiffusive and superdiffusive processes. Some recent examples are: transport in geological formations [2], blinking quantum dots [3], intermittent transport [4,5], wind modeling [6,7], human travel [8], and economics [9]. For many more examples and a good overview over the theoretical background we refer to the reviews [10,11].

Since the CTRW is not Markovian in real time, the joint probability distributions do not factorize. Therefore it is necessary to determine all joint probability distributions to fully characterize the process [12,13]. Montroll and Weiss derived in the aforementioned paper [1] the Montroll-Weiss equation which gives the Fourier-Laplace transform of the one-point distribution. Barkai and Sokolov extended it to a description of the two-point distributions [14]. Another extension to multipoint correlation by Baule and Friedrich [15,16] starts from the description by Fogedby [17] with two independent Langevin equations. Their derivation makes substantial use of the independence between the random processes describing the time and the space evolution. Šanda and Mukamel [12] look at a CTRW on a spatial lattice where the transition probability to another lattice site depends on the current position while the spatial motion is independent of the waiting times. They are mainly considering the case of an external potential such that the transition matrix has a stationary ensemble while aforementioned papers and we consider the situation that temporal and spatial step size is independent of the current position. In this paper we want to present a different method which is essentially an extension of the argument of Montroll and Weiss and which allows to directly write down the joint probability distribution of a possibly space-time coupled CTRW in Fourier-Laplace space. Additionally, we introduce a method to determine the Laplace transforms of the multipoint correlations without having to determine the joint probability distributions first which can become quite complicated when considering several points.

This paper is structured as follows: after introducing some notation, we introduce our method in the next section. We calculate the two and three point probability distribution in Fourier-Laplace space. In Sec. III we demonstrate how to use the general formulation to give a direct proof of the fact [18], that in the unbiased case with finite mean waiting time and in the biased case with finite mean and variance of the waiting time, respectively, the random walk in the scaling limit is Markovian. We proceed by using our method to present an argument by which it is possible to calculate the Laplace transforms of the correlation functions in a direct way, i.e., without determining the joint probability distributions first (Sec. IV). Finally, we apply this method to calculate some correlation functions in the uncoupled and the coupled case.

In this paragraph, we want to introduce some notation. The position of the random walker at time t is denoted by $X(t)$. We assume that at time $t=0$, the walker starts at the origin, i.e., $X(0)=0$. The probability density of doing a step x after a waiting time of t is denoted by $\psi(x,t)$ (we are assuming here the "leaper" [19] type of CTRW, i.e., the spatial movement is done in one leap after waiting while the walker rests during the waiting time, the adaptation to other models is exemplified in Appendix A on the "creeper" [19] model). We denote the distribution of the waiting time by $\phi(t)=\int dx\psi(x,t)$. We adopt the convention from Ref. [14] to distinguish the Fourier or Laplace transform of a function by the naming of the arguments, i.e., the Fourier-Laplace transform of $\psi(x,t)$ will be written as

$$\psi(k,\lambda)=\int dt\int dx e^{-\lambda t+ikx}\psi(x,t). \tag{1}$$

The n-point joint probability density function will be referred to as $p_n(x_1,\ldots,x_n,t_1,\ldots,t_n)$ or short as $p_n(\mathbf{x},\mathbf{t})$. The Fourier-Laplace transform is then

$$p_n(\mathbf{k},\boldsymbol{\lambda})=\int d^nt\int d^nx e^{-\boldsymbol{\lambda}\cdot\mathbf{t}+i\mathbf{k}\cdot\mathbf{x}}p_n(\mathbf{x},\mathbf{t}) \tag{2}$$

with $\boldsymbol{\lambda}\cdot\mathbf{t}=\lambda_1t_1+\cdots+\lambda_nt_n$ and $\mathbf{k}\cdot\mathbf{x}=k_1x_1+\cdots+k_nx_n$. Section II is mainly concerned with the determination of this Fourier-Laplace transform. Especially for the "leaper" model, it will be handy to have the following definition ready

$$\check{p}_n(\mathbf{k},\boldsymbol{\lambda})=\lambda_1\cdots\lambda_np_n(\mathbf{k},\boldsymbol{\lambda}) \tag{3}$$

which implies the following relation in the untransformed variables:

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$$p_n(\mathbf{x},\mathbf{t}) = \int_{0-}^{t_1} dt_1' \cdots \int_{0-}^{t_n} dt_n' \breve{p}_n(\mathbf{x},\mathbf{t}'). \tag{4}$$

II. JOINT PROBABILITY DISTRIBUTIONS

First, we start with splitting up the probability $p_n(\mathbf{x},\mathbf{t})$ into the contributions of the different steps. For non negative integers $q_1,\ldots,q_n$ we call $p_n[q_1,\ldots,q_n](\mathbf{x},\mathbf{t})=p_n[\mathbf{q}](\mathbf{x},\mathbf{t})$ the density $p_n(\mathbf{x},\mathbf{t})$ intersected with the event that t_i is in the waiting time of the (q_i+1)th step (for $i=1,\ldots,n$). Therefore one gets

$$p_n(\mathbf{x},\mathbf{t}) = \sum_{q_1,\ldots,q_n=0}^{\infty} p_n[\mathbf{q}](\mathbf{x},\mathbf{t}). \tag{5}$$

We are now going to factorize this probability distribution in terms describing only one step. We denote by χ_i the random variable describing the step size of the ith step and τ_i the corresponding waiting time. The probability density of the pair (χ_i,τ_i) is given by $\psi(x,t)$ and all (χ_i,τ_i) $(i=1,\ldots)$ are independent random variables. With this notation we can write [24]

$$p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \left\langle \prod_{i=1}^{n} \delta(x_i - \chi_1 - \cdots - \chi_{q_i-1})\,\theta(t_i - \tau_1 - \cdots - \tau_{q_i-1})\,\theta(\tau_1 + \cdots + \tau_{q_i} - t_i) \right\rangle, \tag{6}$$

where δ is the Dirac delta function, θ the Heaviside step function, and $\langle\cdots\rangle$ denotes the expectation value. We now introduce the auxiliary function $(j=1,\ldots)$

$$\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \prod_{i:q_i<j} \delta(x_i)\,\delta(t_i) \prod_{i:q_i\geqslant j} \delta(x_i - \chi_j - \cdots - \chi_{q_i-1}) \times \theta(t_i - \tau_j - \cdots - \tau_{q_i-1})\,\theta(\tau_j + \cdots + \tau_{q_i} - t_i). \tag{7}$$

For $j=1$ we regain Eq. (6) by taking the expectation value

$$p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \langle \Delta_n^{(1)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \rangle \tag{8}$$

while for j larger than every q_i, $\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ collapses to

$$\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \prod_{i=1}^{n} \delta(x_i)\,\delta(t_i) \tag{9}$$

and does not depend any more on the (χ_i,τ_i). In general $\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ only depends on the (χ_i,τ_i) with $i\geqslant j$.

The idea is now to find a function $\eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ which depends only on the random variable (χ_j,τ_j) and fulfills the recursion relation

$$\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \star \Delta_n^{(j+1)}[\mathbf{q}](\mathbf{x},\mathbf{t}), \tag{10}$$

where $\star$ denotes the convolution with respect to $x_1,\ldots,x_n$ and $t_1,\ldots,t_n$ (a Fourier convolution for the x_i and a Laplace convolution for the t_i, i.e., we always assume $t_i\geqslant 0$). The following function will do as can easily be checked by inserting in Eq. (10):

$$\eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \prod_{i:q_i=j} \delta(x_i)\,\theta(\tau_j - t_i) \prod_{i:q_i<j} \delta(x_i)\,\delta(t_i) \times \prod_{i:q_i>j} \delta(x_i - \chi_j)\,\delta(t_i - \tau_j). \tag{11}$$

Putting together Eqs. (8) and (10), we get

$$p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \langle \overset{m}{\underset{j=1}{\star}} \eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \rangle, \tag{12}$$

where m is any natural number greater than every q_i. In Fourier-Laplace space, this reads

$$p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) = \left\langle \prod_{j=1}^{\infty} \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle = \prod_{j=1}^{\infty} \langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \rangle, \tag{13}$$

where we have used the fact that the (χ_i,τ_i) are independent and the $\eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ depend only on the random variable (χ_j,τ_j). Since $\eta_n^{(m)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})=1$ for every m greater than every q_i we can any of these m as the upper limit for j in the product. The function $\eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ can be interpreted as the contribution of the jth step. As a side note, Eq. (11) be easily adapted to other models. We exemplify this for the "creeper" model in Appendix A.

Before we proceed to use Eq. (13) to get an explicit expression for $p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$, we take a step back to Eq. (5). Tough we can calculate $p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ we still have to sum over infinitely many of these terms. In the proceeding paragraphs we want to split this sum Eq. (5) into finitely many subsums which are easy to calculate. The idea is to group the "partial" probabilities (i.e., the $p_n[\mathbf{q}](\mathbf{x},\mathbf{t})$ according to the relative ordering of the steps. For example, for two coefficients we get the orderings $q_1<q_2$, $q_1=q_2$ and $q_2<q_1$; for three we get $q_1<q_2<q_3$, $q_1=q_2<q_3$, $q_1<q_2=q_3$ and $q_1=q_2=q_3$ plus the permutations which result in a different relation (in total 13). We now extend the notation $p_n[\cdots](\mathbf{x},\mathbf{t})$ such that when we write the relation instead of the coefficients between the square brackets, we mean that we sum over all q_i which fulfill this relation. In other words, we intersect the probability with the event that this relation is fulfilled. For example, we write

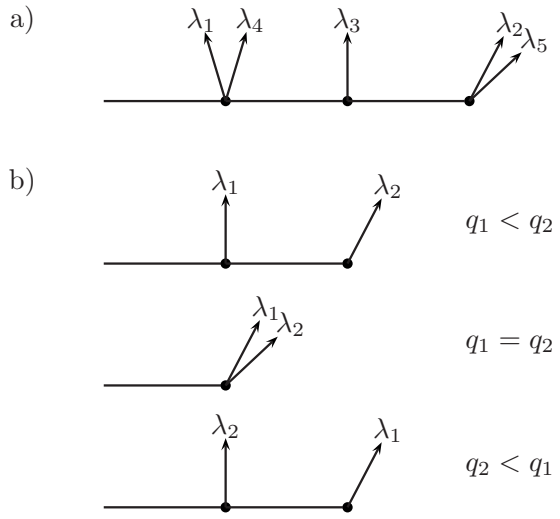

FIG. 1. (a) Example of a diagram appearing for a five point density. Explanations are given in the text. (b) The three diagrams for the two point density. The corresponding relations of the step numbers are given in the right column.

$$p_2[q_1 < q_2](\mathbf{x},\mathbf{t}) = \sum_{\substack{q_1,q_2=0 \\ q_1<q_2}}^{\infty} p_2[q_1,q_2](\mathbf{x},\mathbf{t}) \tag{14}$$

and

$$p_2[q_1 = q_2](\mathbf{x},\mathbf{t}) = \sum_{q_1=0}^{\infty} p_2[q_1,q_1](\mathbf{x},\mathbf{t}). \tag{15}$$

This splits the sum into finitely many different parts. As an example $p_2(\mathbf{x},\mathbf{t})$ can be written as

$$p_2(\mathbf{x},\mathbf{t}) = p_2[q_1 < q_2](\mathbf{x},\mathbf{t}) + p_2[q_1 = q_2](\mathbf{x},\mathbf{t}) + p_2[q_2 < q_1](\mathbf{x},\mathbf{t}), \tag{16}$$

respectively, for $p_2(\mathbf{k},\boldsymbol{\lambda})$

$$p_2(\mathbf{k},\boldsymbol{\lambda}) = p_2[q_1 < q_2](\mathbf{k},\boldsymbol{\lambda}) + p_2[q_1 = q_2](\mathbf{k},\boldsymbol{\lambda}) + p_2[q_2 < q_1](\mathbf{k},\boldsymbol{\lambda}). \tag{17}$$

We will show that for each of these contributions, the sum can be evaluated easily. We found it more intuitive to use little diagrams which represent the relative orderings. One example for a five point function can be seen in Fig. 1(a). The number of steps increases from left to right. A vertex corresponds to a single step in whose waiting time the time parameters of the indices indicated by the outgoing arrows lie. A horizontal line corresponds to any finite number of steps (including zero) taken by the walker without having any time parameter in the step. The example represents $q_1 = q_4 < q_3 < q_2 = q_5$, or using the terminology of the continuous-time random walk, that t_1 and t_4 are in the same step, t_3 in some later, while t_2 and t_5 are together again in any even later step. We do not yet impose a corresponding ordering on the times $t_1,\ldots,t_5$ as this will lead to more complicated terms. We will come later to a more efficient way of using the symmetry under exchange of indices.

We now fix a diagram (respectively, a given ordering). We start with a vertex which corresponds always to a single step of the continuous-time random walk. While in this general frame the exact step number j of the vertex in a diagram can vary, the three types of indices in Eq. (11) are always the same for a given ordering. We use the notation: the set of indices belonging to the times being in a later step than the current one will be denoted by $\mathcal{L}$ ("later," $\mathcal{L}=\{i:q_i>j\}$). The set of of indices belonging to the times belonging to an earlier step will be denoted by $\mathcal{E}$ ("earlier," $\mathcal{E}=\{i:q_i<j\}$). Finally, the set of indices belonging to the times which are in the waiting time of the current time step will be denoted by $\mathcal{V}$ ("vertex," $\mathcal{V}=\{i:q_i=j\}$). In the example Fig. 1(a) we would have for the first vertex $\mathcal{E}=\{\,\}$, $\mathcal{V}=\{1,4\}$ and $\mathcal{L}=\{2,3,5\}$; for the second vertex $\mathcal{E}=\{1,4\}$, $\mathcal{V}=\{3\}$ and $\mathcal{L}=\{2,5\}$; and for the last $\mathcal{E}=\{1,3,4\}$, $\mathcal{V}=\{2,5\}$ and $\mathcal{L}=\{\,\}$. For working with sets of indices the following definitions are helpful:

$$\Lambda_{\mathcal{I}} = \sum_{i\in\mathcal{I}} \lambda_i \quad \text{and} \quad K_{\mathcal{I}} = \sum_{i\in\mathcal{I}} k_i. \tag{18}$$

With this notation we have for a vertex

$$\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})\rangle = \int d\chi \int d\tau \psi(\chi,\tau) \prod_{e\in\mathcal{E}} \delta(x_e)\delta(t_e) \times \prod_{v\in\mathcal{V}} \delta(x_v)\theta(\tau - t_v) \prod_{l\in\mathcal{L}} \delta(x_l - \chi)\delta(t_l - \tau), \tag{19}$$

which is easy to calculate in Fourier-Laplace space:

$$\begin{aligned}\rho_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) &= \langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})\rangle \\ &= \frac{1}{\Pi_{v\in\mathcal{V}}\lambda_v} \int d\chi \int d\tau \psi(\chi,\tau) e^{-\Lambda_{\mathcal{L}}\tau + iK_{\mathcal{L}}\chi} \times \prod_{v\in\mathcal{V}} (1 - e^{-\lambda_v\tau}) \\ &= \frac{1}{\Pi_{v\in\mathcal{V}}\lambda_v} \sum_{\mathcal{J}\in\mathcal{P}(V)} (-1)^{|\mathcal{J}|} \psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}} + \Lambda_{\mathcal{J}}),\end{aligned} \tag{20}$$

where $\mathcal{P}(\mathcal{V})$ is the power set of $\mathcal{V}$ and $|\mathcal{J}|$ denotes the number of elements in $\mathcal{J}$ (the cardinality). The sum in the last line has in the λ argument of ψ the term $\Lambda_{\mathcal{L}}$ plus every combination of λ_j with $j\in\mathcal{J}$ with a positive sign in front of ψ if it is an even number of elements and a negative sign if it is an odd number.

We now proceed to calculate the contribution of a horizontal line. For a given horizontal line all steps have the same types of indices in the sets $\mathcal{E}$ and $\mathcal{L}$ while always $\mathcal{V}=\{\,\}$. Looking again at our example Fig. 1(a): the line before the first vertex has $\mathcal{E}=\{\,\}$ and $\mathcal{L}=\{1,2,3,4,5\}$, the line between the first and second vertex $\mathcal{E}=\{1,4\}$ and $\mathcal{L}=\{2,3,5\}$ while the last line between the second and third vertex has $\mathcal{E}=\{1,3,4\}$ and $\mathcal{L}=\{2,5\}$. By using $\mathcal{V}=\{\,\}$ in Eq. (20), we see that the contribution of a single step inside a horizontal line in the diagram is $\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})$. When we calculate the probability distribution corresponding to a given diagram (respectively, ordering), every nonnegative number of steps in a

horizontal line is possible and we have to add them up. This gives the contribution of a horizontal line

$$\rho_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) = \sum_{r=0}^{\infty} \psi^r(K_{\mathcal{L}},\Lambda_{\mathcal{L}}) = \frac{1}{1-\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})}. \quad (21)$$

We can now put the contributions together. When we are summing the $p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ which constitute the probability belonging to a fixed diagram, the only parameters we have to sum over are all possible numbers of steps in the horizontal lines. Comparing with Eq. (13) one sees that one gets this sum by multiplying the contribution (20) for each vertex and (21) for each line in the diagram. This result can alternatively be derived by using a renewal equation [20]. We show this approach in Appendix B.

The calculation of a joint probability distribution $p_n(\mathbf{k},\boldsymbol{\lambda})$ in Fourier-Laplace space of a continuous-time random walk reduces therefore to the following steps: draw all possible diagrams with n indices (or, equivalently, determine all possible orderings of the n indices). For each diagram multiply the finitely many contributions Eqs. (20) and (21). Adding up these products gives $p_n(\mathbf{k},\boldsymbol{\lambda})$.

The simplest case is of course the one point density which has only one diagram with one line and one vertex. Using the described procedure gives directly the Montroll-Weiss equation [1]

$$p_1(k,\lambda) = \frac{1}{\lambda}\frac{1-\phi(\lambda)}{1-\psi(k,\lambda)}. \quad (22)$$

Our next example is the two point density for which the corresponding diagrams are shown in Fig. 1(b)

$$p_2[q_1<q_2](\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{\lambda_1}\frac{\psi(k_2,\lambda_2)-\psi(k_2,\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)} \times \frac{1}{\lambda_2}\frac{1-\phi(\lambda_2)}{1-\psi(k_2,\lambda_2)},$$

$$p_2[q_1=q_2](\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{\lambda_1\lambda_2}\frac{1-\phi(\lambda_1)-\phi(\lambda_2)+\phi(\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)},$$

$$p_2[q_2<q_1](\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{\lambda_2}\frac{\psi(k_1,\lambda_1)-\psi(k_1,\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)} \times \frac{1}{\lambda_1}\frac{1-\phi(\lambda_1)}{1-\psi(k_1,\lambda_1)}, \quad (23)$$

where we have used $\phi(\lambda)=\psi(0,\lambda)$. In combination with Eq. (17) this reproduces a result obtained by Barkai and Sokolov [14].

Simplifications. Till now, we have introduced a method which allows to write down directly the Fourier-Laplace transform of the n-point joint probability distribution. But it is clear, that these terms become very large, e.g., already for $p_3(\mathbf{k},\boldsymbol{\lambda})$ we would have to consider 13 diagrams. But the joint probability distribution are symmetric under exchange of the indices which is a property which can also be applied to the "partial" probability associated with a diagram. If we look at Eq. (23), it is easy to see that the situations $q_1<q_2$ and $q_2<q_1$ emerge from each other by permutation of the indices while the case $q_1=q_2$ is symmetric in the indices. In the rest of this section, we will show how to use this symmetry to ease the calculation.

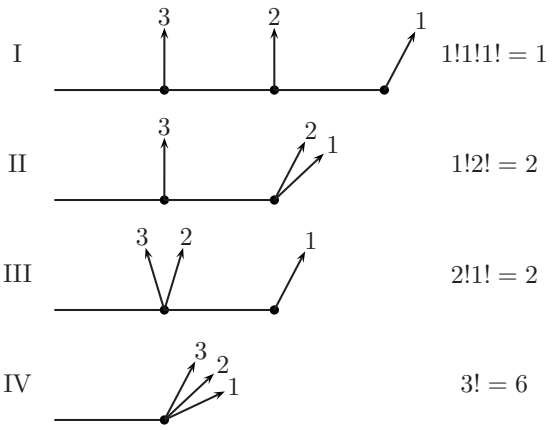

FIG. 2. The diagrams relevant for the calculation of the three point probability density. To indicate that this are only the ones with sorted indices, the ends of the arrows are not attached to variables but just labeled with decreasing numbers. The right column gives the multiplicity with which this diagram is counted.

In more detail, we are looking for a $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ which gives $p_n(\mathbf{k},\boldsymbol{\lambda})$ via

$$p_n(\mathbf{k},\boldsymbol{\lambda}) = \sum_{\pi\in S_n} \omega_n(k_{\pi(1)},\ldots,k_{\pi(n)},\lambda_{\pi(1)},\ldots,\lambda_{\pi(n)}), \quad (24)$$

where S_n is the symmetric group of n elements, i.e., the sum runs over all permutations π of the indices. Ideally, $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ will contain every type of diagram only once (e.g., for the two-point probability $q_1<q_2$ and $q_2<q_1$ are different contributions, but the diagrams have the same type or form). To achieve this, we draw all different structures of diagrams that may appear only once and number the outgoing arrows from n down to 1 (we do not put the λ_i at the arrows any more to emphasize that it is not connected to a specific index but an abstract numbering). For the case $n=3$ this is shown in Fig. 2. In principle, we can now apply the same rules to construct the $p_n(\mathbf{k},\boldsymbol{\lambda})$ to get the $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ by summing over the different types of diagrams. We only have to keep track that we sum in Eq. (24) over diagrams several times when they have vertices with more than one emerging arrow—namely, taken the factorial of the number of arrows for each vertex. Therefore one has to divide by this symmetry factor. In Fig. 2 these numbers are given in the right column. Therefore one possible way to describe the contributions for $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ is

$$\rho^{\omega}_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{|\mathcal{V}|!}\frac{1}{\Pi_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(V)}(-1)^{|\mathcal{J}|}\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}}+\Lambda_{\mathcal{J}}) \quad (25)$$

for each vertex while the contribution of a line stays the same. This form has the advantage that it one directly generalize it to other models then the "leaper" model and is useful in some limit considerations shown in Sec. III.

If one uses the "leaper" model of the continuous-time random walk, we consider in general an alternative form as being more useful which uses specifics of this model. Similarly to Eq. (3), we can factor out the $\frac{1}{\lambda_i}$ which gives rise to the definition

$$\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda}) = \lambda_1 \cdots \lambda_n \omega_n(\mathbf{k},\boldsymbol{\lambda}), \tag{26}$$

and we have

$$\breve{p}_n(\mathbf{k},\boldsymbol{\lambda}) = \sum_{\pi \in S_n} \breve{\omega}_n(k_{\pi(1)}, \ldots, k_{\pi(n)}, \lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)}). \tag{27}$$

The form which is derived here will additionally allow a direct interpretation of $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ and fix it completely.

The first change is that we now consider $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ instead of $\omega_n(\mathbf{k},\boldsymbol{\lambda})$. This is achieved by removing the prefactor $(\Pi_{v \in \mathcal{V}} \lambda_v)^{-1}$ from the definition of a vertex. Second, we notice that—for a given diagram—we can apply any permutation on the indices $\mathcal{V}$ of any vertex without changing anything in the factors coming from horizontal lines of other vertices. We use this freedom to apply an permutation on every element of the sum Eq. (25) such that it can be written solely with the combinations $K_q = K_{\{1,\ldots,q\}}$ and $\Lambda_q = \Lambda_{\{1,\ldots,q\}}$. This is possible because of the decreasing numbering and gives an alternative form for the vertex [now as a contribution to $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$]

$$\nu(d,h) = \frac{1}{d!} \sum_{j=0}^{d} \binom{d}{j} (-1)^j \psi(K_{h-d}, \Lambda_{h-d+j}) \tag{28}$$

which takes as arguments the degree of the vertex d (i.e., the number of arrows leaving) and the first (and therefore highest) index h. Now, applying any permutation to the indices either leaves the term invariant or transforms it to one which uses other combination of argument other than K_q or Λ_q ($q = 1, \ldots$). Therefore it is not possible that different summands in Eq. (27) cancel in whole or part which is useful in connection with computer algebra systems.

At a first glance, the restriction to the arguments K_q and Λ_q can be seen as a method simply to reduce the number of possible arguments. But it is possible to give an interpretation of this representation. To see this, we start with a function $f(t_1, \ldots, t_n)$ in the variables $t_1, \ldots, t_n \geqslant 0$. The corresponding parameters of the Laplace transform are denoted as usual by $\lambda_1, \ldots, \lambda_n$. Now, if $f(\mathbf{t})$ has support contained in the domain $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n$, it is possible to change variables to $(t_1 - t_2), \ldots, (t_n - t_{n-1}), t_n \geqslant 0$. Carrying out this variable transform in the Laplace transform, one sees from the identity

$$\lambda_1 t_1 + \cdots + \lambda_n t_n = \Lambda_1 (t_1 - t_2) + \cdots + \Lambda_{n-1}(t_{n-1} - t_n) + \Lambda_n t_n \tag{29}$$

that the Laplace parameters to these variables are just the $\Lambda_1, \ldots, \Lambda_n$. Conversely, if we write the Laplace transform $f(\lambda)$ in the variables $\Lambda_1, \ldots, \Lambda_n$ and if we know that it is a Laplace transform in these variables then we can conclude that $f(\mathbf{t})$ has support in the domain $t_1 \geqslant \cdots \geqslant t_n$. Now, if we build $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ from the contributions Eq. (28), we can expand the products and end with summands which consist of factors depending only on one Λ_i. These are either of the form $\psi(K_j, \Lambda_i)$ or $\psi(K_j, \Lambda_i)/[1 - \psi(K_i, \Lambda_i)]$. For both it is then clear from construction that they constitute a Laplace transform. Therefore, we know that $\breve{\omega}_n(\mathbf{x},\mathbf{t})$ vanishes outside of $t_1 \geqslant \cdots \geqslant t_n$ if we use the form Eq. (28) for the vertex. In reverse, putting this condition on $\breve{\omega}_n(\mathbf{x},\mathbf{t})$ would have fixed $\breve{\omega}_n(\mathbf{x},\mathbf{t})$ under all functions which satisfy Eq. (27).

Applying this to the three point function with the diagrams depicted in Fig. 2 one gets the contributions

$$\begin{aligned}
\breve{\omega}_3^{\mathrm{I}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{\nu(1,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(1,2)}{1-\psi(K_2,\Lambda_2)} \frac{\nu(1,1)}{1-\psi(K_1,\Lambda_1)} \\
&= \frac{\psi(K_2,\Lambda_2) - \psi(K_2,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \\
&\quad \times \frac{\psi(K_1,\Lambda_1) - \psi(K_1,\Lambda_2)}{1-\psi(K_2,\Lambda_2)} \frac{1-\phi(\Lambda_1)}{1-\psi(K_1,\Lambda_1)},
\end{aligned}$$

$$\begin{aligned}
\breve{\omega}_3^{\mathrm{II}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{\nu(1,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(2,1)}{1-\psi(K_2,\Lambda_2)} \\
&= \frac{\psi(K_2,\Lambda_2) - \psi(K_2,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \times \frac{1}{2} \frac{1 - 2\phi(\Lambda_1) + \phi(\Lambda_2)}{1-\psi(K_2,\Lambda_2)},
\end{aligned}$$

$$\begin{aligned}
\breve{\omega}_3^{\mathrm{III}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{\nu(2,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(1,1)}{1-\psi(K_1,\Lambda_1)} \\
&= \frac{1}{2} \frac{\psi(K_1,\Lambda_1) - 2\psi(K_1,\Lambda_2) + \psi(K_1,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \\
&\quad \times \frac{1-\phi(\Lambda_1)}{1-\psi(K_1,\Lambda_1)}
\end{aligned}$$

$$\begin{aligned}
\breve{\omega}_3^{\mathrm{IV}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{\nu(3,3)}{1-\psi(K_3,\Lambda_3)} \\
&= \frac{1}{6} \frac{1 - 3\phi(\Lambda_1) + 3\phi(\Lambda_2) - \phi(\Lambda_3)}{1-\psi(K_3,\Lambda_3)},
\end{aligned}$$

$$\breve{\omega}_3(\mathbf{k},\boldsymbol{\lambda}) = \breve{\omega}_3^{\mathrm{I}}(\mathbf{k},\boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{II}}(\mathbf{k},\boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{III}}(\mathbf{k},\boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{IV}}(\mathbf{k},\boldsymbol{\lambda}). \tag{30}$$

It is possible to formulate the contributions to $\breve{\omega}_n$ in a recursive way. For this one notices that for each diagram, leaving out the factor stemming from the first line and the first vertex, gives exactly the same contribution as the diagram (with less indices) with this line and vertex removed. Putting this into formulas, gives

$$\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda}) = \sum_{d=1}^{n} \frac{\nu(d,n)}{1-\psi(\mathbf{k},\boldsymbol{\lambda})} \breve{\omega}_{n-d}(\mathbf{k},\boldsymbol{\lambda}). \tag{31}$$

The $\breve{\omega}_{n-d}$ takes of course only $(n-d)$ k and λ arguments, but with the decreasing numbering introduced above, these are the first $(n-d)$ ones.

III. SCALING LIMIT WITH FINITE MEAN WAITING TIME

In this section we want to give an example of how to use the method introduced in the last section in a more general

setting. We want to look at the scaling limit in case the waiting time distribution has a finite mean (or additionally a finite second moment, depending on the definition used when the mean of the spatial step size distribution does not vanish). We are going to show that under these conditions all finite point joint probability distributions become Markovian in the limit. The result itself is not new (e.g., see Dentz and Berkowitz [18]), but we think that it is nevertheless instructive to look at an argument using the introduced method.

When considering stochastic processes one is often interested in the long time behavior. A special class introduced under the name "semi-stable" processes in a slightly broader setting by Lamperti [21] are the ones appearing as limit processes in the scaling limit. The limiting process is

$$X^{\lim}(t) = \zeta^{\alpha} X\left(\frac{t}{\zeta}\right) \quad \text{for } \zeta \to 0, \tag{32}$$

where α is the scaling exponent. The limit used for the stochastic process means convergence in distribution of all finite point joint probability distributions. The exponent α is fixed such that it is the largest one without a trivial limit process [i.e., $X(t)=0$ for all $t \geqslant 0$]. In general it is possible that one has to extend the power law behavior by a slowly varying function [20], but we will assume for simplicity that this is not the case here. Many processes converge to Brownian motion with $\alpha=1/2$. Writing the convergence conditions for the probability density, yields the limit as

$$p_n^{\lim}(\mathbf{x},\mathbf{t}) = \lim_{\zeta\to 0} \frac{1}{\zeta^{n\alpha}} p_n\left(\frac{1}{\zeta^{\alpha}}\mathbf{x}, \frac{1}{\zeta}\mathbf{t}\right). \tag{33}$$

Going to Fourier-Laplace space and the more convenient $\check{p}_n$, gives

$$\check{p}_n^{\lim}(\mathbf{k},\boldsymbol{\lambda}) = \lim_{\zeta\to 0} \check{p}_n(\zeta^{\alpha}\mathbf{k}, \zeta\boldsymbol{\lambda}) \tag{34}$$

as the limit we are interested in with α chosen to be the largest value with this object still depending non trivially on $\mathbf{k}$.

Now, we focus on the case that the waiting time distribution has the finite mean value τ and spatial step distribution has zero mean. To avoid distracting technicalities, we will restrict ourselves to the space-time independent case where the spatial step has variance σ^2 which will lead to Brownian motion. The argument generalizes straightforwardly to other settings. Our assumptions give rise to the following asymptotic behavior of the Fourier-Laplace transform around $\lambda, k \to 0$

$$\phi(\lambda) = 1 - \tau\lambda + o(\lambda) \tag{35}$$

and

$$\psi(k,\lambda) = \phi(\lambda)\left(1 - \frac{\sigma^2}{2}k^2 + o(k^2)\right). \tag{36}$$

where we use the Landau notation. The scaling exponent is $\alpha=1/2$. Let us first look at the denominators stemming from the lines. Its contributions are

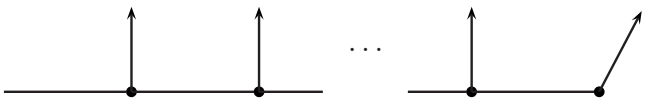

FIG. 3. This diagrams represents the Markovian contribution of the random walk. It is the only one that survives in the scaling limit with finite mean waiting time.

$$\rho_{\text{line}}^{-1}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) = 1 - \psi(\zeta^{1/2}K_i, \zeta\Lambda_i) = \zeta\left(\tau\Lambda_i + \frac{\sigma^2}{2}K_i^2\right) + o(\zeta). \tag{37}$$

Now we move on to the contribution of a vertex. The $\mathbf{k}$ contribution factors out by our assumptions and becomes unity in the scaling limit (which in retrospect justifies the scaling exponent because it is therefore determined purely by the behavior of the denominators). A degree one vertex gives the contribution

$$\rho_{\substack{\text{vertex}\\ \deg=1}}^{\omega}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) = \phi(\zeta\Lambda_{i-1}) - \phi(\zeta\Lambda_i) + o(\zeta) = \zeta\tau\lambda_i + o(\zeta). \tag{38}$$

For the vertices of higher degree we apply the definition used in Eq. (29)

$$\begin{aligned}\rho_{\substack{\text{vertex}\\ \deg\geqslant 2}}^{\omega}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) &= \sum_{\mathcal{J}\in\mathcal{P}(V)} (-1)^{|\mathcal{J}|}\phi[\zeta(\Lambda_{\mathcal{L}} + \Lambda_{\mathcal{J}})] + o(\zeta)\\ &= \zeta\tau \sum_{\mathcal{J}\in\mathcal{P}(V)} (-1)^{|\mathcal{J}|}\Lambda_{\mathcal{J}} + o(\zeta)\\ &= o(\zeta).\end{aligned} \tag{39}$$

Since we have the same number of lines and vertices, all terms with vertices of degree 2 or higher vanish in the scaling limit. Therefore only the diagram Fig. 3 survives. The interpretation is that we can neglect the case that two or more times are in the same step. The scaling limit gives

$$\check{\omega}_n^{\lim}(\mathbf{k},\boldsymbol{\lambda}) = \frac{\tau\lambda_n}{\tau\Lambda_n + \frac{\sigma^2}{2}K_n} \cdots \frac{\tau\lambda_1}{\tau\Lambda_1 + \frac{\sigma^2}{2}K_1}. \tag{40}$$

With

$$p_1^{\lim}(x,t) = \sqrt{\frac{\tau}{2\pi\sigma^2}\frac{1}{t}}\exp\left(-\frac{\tau}{2\sigma^2}\frac{x^2}{t}\right) \tag{41}$$

being the inverse Fourier-Laplace transform of

$$p_1^{\lim}(k,\lambda) = \frac{1}{\lambda + \frac{\sigma^2}{2\tau}k^2} \tag{42}$$

one gets for $p_n^{\lim}(\mathbf{x},\mathbf{t})$ with $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{n-1} \geqslant t_n$

$$\begin{aligned}p_n^{\lim}(\mathbf{x},\mathbf{t}) = p_1^{\lim}(x_n,t_n)p_1^{\lim}(x_{n-1}-x_n, t_{n-1}-t_n)\\ \times \cdots p_1^{\lim}(x_1 - x_2, t_1 - t_2)\end{aligned} \tag{43}$$

which is exactly the multi point density of Brownian motion.

Let us now have a look at the case that the spatial step distribution has a finite mean $\mu \neq 0$. Using the definition Eq.

(32) with $\alpha=1$ and the same arguments gives then

$$p_n^{\lim}(\mathbf{x},\mathbf{t}) = \prod_{i=1}^{n} \delta\left(x_i - \frac{\mu}{\tau} t_i\right), \tag{44}$$

which corresponds to a uniform deterministic motion with velocity $\frac{\mu}{\tau}$. This is of course Markovian but not a stochastic process in the strict sense. To see a possible diffusion on top of the drift, we define the process

$$Y(t) = X(t) - \frac{\mu}{\tau} t \tag{45}$$

with the joint probability distributions in Fourier-Laplace space

$$p_n^{Y}(\mathbf{k},\boldsymbol{\lambda}) = p_n\left(\mathbf{k}, \boldsymbol{\lambda} - i\frac{\mu}{\tau}\mathbf{k}\right), \tag{46}$$

where $p_n(\mathbf{k},\boldsymbol{\lambda})$ still denotes the joint probability distributions of the process $X(t)$. The scaling limit of this process can be calculated as

$$p_n^{Y,\lim} = \lim_{\zeta\to 0} \zeta^n p_n\left(\zeta^\alpha \mathbf{k}, \zeta\boldsymbol{\lambda} - i\frac{\mu}{\tau}\zeta^\alpha \mathbf{k}\right). \tag{47}$$

If we want this to converge to a Markovian process (with $\alpha=\frac{1}{2}$), we additionally need the existence of the second moment of the waiting time distribution to account for the different orders of ζ appearing in the second argument. But if we have this condition, we can essentially use the same arguments as above to show that the process is Markovian.

IV. MULTIPOINT CORRELATIONS

When on wants to compare a stochastic model with measured numerical data, one often does not look at the full probability distributions. This can have several reasons: estimating probability distributions needs normally a lot of data which is not always available or in the case of the continuous-time random walk the analytical expressions for the joint probability distributions become large quite fast [e.g., see Eqs. (23) and (30)]. Therefore one is often interested in the multi point correlations which are defined by

$$C_n(\mathbf{t}) = \langle X(t_1)\cdots X(t_n)\rangle \tag{48}$$

or, in Laplace space,

$$C_n(\boldsymbol{\lambda}) = \left.\frac{\partial}{i\partial k_1}\cdots\frac{\partial}{i\partial k_n} p_n(\mathbf{k},\boldsymbol{\lambda})\right|_{\mathbf{k}=\mathbf{0}}. \tag{49}$$

Equation (49) can be applied to any joint probability distribution determined with the method introduced in Sec. II. This works with any model of a continuous-time random walk but it has the disadvantage of having to determine the joint probability distribution first. In this section we want to focus on the "leaper" model and introduce a method which allows us to write down the Laplace transform of the multi point correlations without having to determine the joint probabilities first.

Similar to the case of probability densities, it is convenient to have the definitions

$$\breve{C}_n(\boldsymbol{\lambda}) = \lambda_1\cdots\lambda_n C_n(\boldsymbol{\lambda}) \tag{50}$$

which gives in the untransformed variables

$$C_n(\mathbf{t}) = \int_{0-}^{t_1} dt_1' \int_{0-}^{t_n} dt_n' \cdots \breve{C}_n(\mathbf{t}'). \tag{51}$$

Additionally, we define

$$\breve{\gamma}_n(\boldsymbol{\lambda}) = \left.\frac{\partial}{i\partial k_1}\cdots\frac{\partial}{i\partial k_n}\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\right|_{\mathbf{k}=\mathbf{0}}, \tag{52}$$

such that

$$\breve{C}_n(\boldsymbol{\lambda}) = \sum_{\pi\in S_n} \breve{\gamma}_n(\lambda_{\pi(1)},\ldots,\lambda_{\pi(n)}). \tag{53}$$

In the remainder we will use $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ by constructing it with the definition for the vertex given in Eq. (28), i.e., the support of $\breve{\omega}_n(\mathbf{x},\mathbf{t})$ is contained in the domain $t_n \leqslant \ldots \leqslant t_2 \leqslant t_1$ (i.e., we can use the Λ_i as the natural variables of the Laplace transform). This gives directly that the support of $\breve{\gamma}_n(\mathbf{t})$ is also contained in $t_n \leqslant \cdots \leqslant t_2 \leqslant t_1$.

We can see from Eq. (49) that the multi point correlations do not depend on the full spatial-temporal probability distribution of the steps but only the first n derivatives with respect to k. This suggests the definition

$$\phi_q(\lambda) = \left.\left(\frac{\partial}{i\partial k}\right)^q \psi(k,\lambda)\right|_{k=0}. \tag{54}$$

We have $\phi_0(\lambda)=\phi(\lambda)$ as the marginal distribution of the waiting times, while $\phi_q(\lambda)$ can be interpreted as the Laplace transform of the qth moment of the spatial jump distribution depending on the waiting time.

We now proceed to calculate $\breve{\gamma}_n(\boldsymbol{\lambda})$. To do this we notice that for any diagram contributing to $\breve{\gamma}_n(\boldsymbol{\lambda})$ the variable k_n does only appear in the contribution of the first horizontal line. This can also be seen from Eq. (31) by noting that $K_n = k_1 + \cdots + k_n$ is the only sum of the $K_i = k_1 + \cdots + k_i$ which contains k_n. Therefore multiplying Eq. (31) with $[1 - \psi(K_n,\Lambda_n)]$ leaves us with an equation for which the right-hand side is now independent of k_n. This gives

$$\frac{\partial}{i\partial k_n}[1 - \psi(K_n,\Lambda_n)]\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda}) = 0. \tag{55}$$

Taking additionally the derivatives with respect to $k_1,\ldots,k_{n-1}$ therefore leads to

$$\left.\frac{\partial^n}{i^n \partial k_{\{1,\ldots,n\}}}[1 - \psi(K_n,\Lambda_n)]\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\right|_{\mathbf{k}=\mathbf{0}} = 0, \tag{56}$$

where we introduced the notation

$$\frac{\partial^{|\mathcal{J}|}}{\partial k_{\mathcal{J}}} \tag{57}$$

to describe the differentiation with respect to all k_i for which the index i is in the set $\mathcal{J}$. Applying the product rule of differentiation gives

$$[1-\phi_0(\Lambda_n)]\breve{\gamma}_n(\mathbf{\lambda}) = \sum_{d=1}^{n} \phi_d(\Lambda_n) \sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\}) \\ |\mathcal{J}|=n-d}} \frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}} \breve{\omega}_n(\mathbf{k},\mathbf{\lambda})\Big|_{\mathbf{k}=\mathbf{0}}, \tag{58}$$

where the second sum runs over all subsets of $\{1,\ldots,n\}$ with a given cardinality. In other words, it is a sum over all differential operators of a given order which can be built by the differentiation with respect to $k_1,\ldots,k_n$ with no second or higher order differentiation with respect to a single k_i.

We will show in Appendix C that we can evaluate these operators with the equality (actually, we derive there a more general form, allowing for higher derivatives)

$$\sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\}) \\ |\mathcal{J}|=n-d}} \frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}} \breve{\omega}_n(\mathbf{k},\mathbf{\lambda})\Big|_{\mathbf{k}=\mathbf{0}} = \frac{1}{d!}\breve{\gamma}_{n-d}(\mathbf{\lambda}). \tag{59}$$

Since the derivation of Eq. (59) is a simple but purely technical handling of indices, we will only give some ideas which make Eq. (59) plausible, but leave the details to the appendix.

Assume we would have one of these operators of order $(n-d)$ acting on $\breve{p}_n(\mathbf{k},\mathbf{\lambda})$, i.e., given $\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\})$ with $|\mathcal{J}|=(n-d)$ we look at

$$\frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}} \breve{p}_n(\mathbf{k},\mathbf{\lambda})\Big|_{\mathbf{k}=\mathbf{0}}. \tag{60}$$

By construction this corresponds to a $(n-d)$-point correlation, i.e., it can be expressed by $\breve{C}_{n-d}(\cdots)$ (dropping the arguments with indices which are not contained in $\mathcal{J}$). Now, symmetrizing $\breve{\omega}_n(\mathbf{k},\mathbf{\lambda})$ gives $\breve{p}_n(\mathbf{k},\mathbf{\lambda})$, while in Eq. (58) we have the reversed situation: we consider $\breve{\omega}_n(\mathbf{k},\mathbf{\lambda})$ with a symmetrized differential operator acting on it. Since we can distinguish the different permutations of $\breve{\omega}_n(\cdots)$ by their support, it is likely that we can identify the result as a multiple of $\breve{\gamma}_{n-d}(\mathbf{\lambda})$.

Putting Eq. (59) into Eq. (58) becomes

$$\breve{\gamma}_n(\mathbf{\lambda}) = \sum_{d=1}^{n} \frac{1}{d!}\frac{\phi_d(\Lambda_n)}{1-\phi(\Lambda_n)}\breve{\gamma}_{n-d}(\mathbf{\lambda}) \tag{61}$$

with the natural definition $\breve{\gamma}_0=1$. Again we take up the convention that the $(n-d)$ arguments of $\breve{\gamma}_{n-d}(\mathbf{\lambda})$ are the first λ_i's, i.e., $\breve{\gamma}_{n-d}(\mathbf{\lambda})=\breve{\gamma}_{n-d}(\lambda_1,\ldots,\lambda_{n-d})$.

Equation (61) is the main result of this section. Applying it gives

$$\breve{\gamma}_1(\mathbf{\lambda}) = \frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)}, \tag{62}$$

$$\breve{\gamma}_2(\mathbf{\lambda}) = \frac{\phi_1(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{2}\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)}, \tag{63}$$

$$\breve{\gamma}_3(\mathbf{\lambda}) = \frac{\phi_1(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_1(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{2}\frac{\phi_1(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)} + \frac{1}{2}\frac{\phi_2(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{6}\frac{\phi_3(\Lambda_3)}{1-\phi(\Lambda_3)}. \tag{64}$$

We now proceed to translate the results back from Laplace space to the real time. For this we define $g_q(t)$ as the inverse Laplace transform of

$$\frac{\phi_q(\lambda)}{1-\phi(\lambda)} \tag{65}$$

for $t\geqslant 0$ and zero otherwise. The first two examples Eqs. (62) and (63) are in real time

$$\breve{\gamma}_1(t_1) = g_1(t_1),$$

$$\breve{\gamma}_2(t_1,t_2) = g_1(t_2)g_1(t_1-t_2) + \frac{1}{2}g_2(t_2)\delta(t_1-t_2). \tag{66}$$

To finally get the n-point correlation function $C_n(\mathbf{t})$, one still has to perform an integration. In accordance with the notation used before, we call this integration of $\breve{\gamma}_n(\mathbf{t})$ by $\gamma_n(\mathbf{t})$ and we have

$$\gamma_n(\mathbf{\lambda}) = \frac{1}{\lambda_1\cdots\lambda_n}\breve{\gamma}_n(\mathbf{\lambda}), \tag{67}$$

where we get $C_n(\mathbf{t})$ just by summing over all permutations of the indices of $\gamma_n(\mathbf{t})$ [similar to Eq. (53)]. The evaluation of $\gamma_n(\mathbf{t})$ can always be reduced to arguments in the domain to $t_1\geqslant t_2\geqslant\cdots\geqslant t_n$ via

$$\gamma_n(\mathbf{t}) = \gamma_n(\mathbf{t}') \tag{68}$$

with $t_j'=\min\{t_1,t_2,\ldots,t_j\}$. These two terms are equal since the difference is an integration over a domain in which $\breve{\gamma}_n(\mathbf{t})$ vanishes. As example, for $t_1\geqslant t_2$ we have

$$C_2(t_1,t_2) = \gamma_2(t_1,t_2) + \gamma_2(t_2,t_1) = \gamma_2(t_1,t_2) + \gamma_2(t_2,t_2). \tag{69}$$

In the domain $t_1\geqslant t_2\geqslant\cdots\geqslant t_n$ the evaluation of $\gamma_n(\mathbf{t})$ goes over to the integral

$$\gamma_n(\mathbf{t}) = \int_{0-}^{t_n} d\tau_n \int_{\tau_n-}^{t_{n-1}} d\tau_{n-1}\cdots\int_{\tau_2-}^{t_1} d\tau_1\,\breve{\gamma}_n(\boldsymbol{\tau}). \tag{70}$$

The minus sign behind the lower bounds expresses that a δ function sitting at the boundary [e.g., a $\delta(\tau_{i+1}-\tau_i)$] is fully evaluated [which is in the definition of the Laplace transform and which can be seen in the fact that the Laplace transform of $\delta(t)$ is simply 1]. Before we come to some examples how to evaluate this expression in the long time limit, we yet want to give two small remarks.

Evaluating this integral (70) with two or more t_i being equal, we have a convolution which we can also express directly in Laplace space. Taking additionally the δ function into account, Eq. (69) is in Laplace space

$$\tilde{C}_2(\lambda_1,\lambda_2)=\frac{1}{\lambda_1\lambda_2}\frac{\phi_1(\lambda_1+\lambda_2)}{1-\phi(\lambda_1+\lambda_2)}\frac{\phi_1(\lambda_1)}{1-\phi(\lambda_1)}+\frac{1}{\lambda_1\lambda_2}\frac{\phi_2(\lambda_2)}{1-\phi(\lambda_2)}+\frac{1}{\lambda_1\lambda_2}\left(\frac{\phi_1(\lambda_2)}{1-\phi(\lambda_2)}\right)^2. \tag{71}$$

This is not the complete two point correlation, but $C_2(t_1,t_2)$ and $\tilde{C}_2(t_1,t_2)$ coincide for $t_1 \geqslant t_2$.

When the correlation function is not symmetric under exchange of the times [e.g., we take different powers of the $X(t_i)$] the corresponding differential operator in Fourier space is also not symmetric unter permutations of the indices and it is disadvantageous to work directly with the $\breve{\gamma}_n(\boldsymbol{\lambda})$. In this case it is easier to use $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ by noting that $[1-\psi(K_n,\Lambda_n)]\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ is a sum of terms which are independent of at least one k_i and therefore differentiating with respect to each k_i $(i=1,\ldots,n)$ at least once will yield a zero.

A. The uncoupled case

We assume here that the step size distribution is independent from the waiting time distribution, i.e., $\phi_q(\lambda)=\mu_q\phi(\lambda)$ where μ_q is the qth moment of the step size distribution. We further assume that the waiting time distribution is in the normal domain of attraction of an one-sided Lévy-distribution with exponent β $(0<\beta<1)$, i.e., $\phi(\lambda)=1-(\tau_0\lambda)^\beta+o(\lambda^\beta)$ where τ_0 is the time constant [20]. The asymptotic results in this subsection for the uncoupled case can also be obtained by applying the method by Baule and Friedrich [15] which we therefore use as a consistency check.

As an example we assume a uncoupled CTRW with a step size distribution with vanishing mean (i.e., $\mu_1=0$). The four-point correlation is then given by

$$\breve{\gamma}_4(\boldsymbol{\lambda})=\frac{\mu_2^2}{4}\frac{\phi(\Lambda_4)}{1-\phi(\Lambda_4)}\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}+\frac{\mu_4}{24}\frac{\phi(\Lambda_4)}{1-\phi(\Lambda_4)}\simeq\frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{1}{\Lambda_4^\beta}\frac{1}{\Lambda_2^\beta}, \tag{72}$$

where we use "≃" for the long time behavior (corresponding to small λ).

We get for $t_1 \geqslant t_2 \geqslant t_3 \geqslant t_4$

$$\gamma_4(\mathbf{t})\simeq\frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{1}{\Gamma(\beta)^2}\int_0^{t_4}ds_4\int_{s_4}^{t_2}ds_2 s_4^{\beta-1}(s_2-s_4)^{\beta-1}=\frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{t_2^\beta t_4^\beta}{\Gamma(\beta+1)^2}\beta\int_0^1 ds s^{\beta-1}\left(1-\frac{t_4}{t_2}s\right)^\beta. \tag{73}$$

The last integral (with the prefactor β) is an integral representation of the hypergeometric function $F(a,b;c;z)$ [[22] Eq. (15.3.1)]. Therefore, we can write

$$\gamma_4(\mathbf{t})\simeq\frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{t_2^\beta t_4^\beta}{\Gamma(\beta+1)^2}F\left(\beta,-\beta;1+\beta;\frac{t_4}{t_2}\right). \tag{74}$$

Summing over the permutations gives

$$C_4(\mathbf{t})\simeq\frac{\mu_2^2}{\tau_0^{2\beta}}\left[\frac{t_4^\beta t_2^\beta}{\Gamma(\beta+1)^2}F\left(\beta,-\beta;1+\beta;\frac{t_4}{t_2}\right)+2\frac{t_4^\beta t_3^\beta}{\Gamma(\beta+1)^2}F\left(\beta,-\beta;1+\beta;\frac{t_4}{t_3}\right)+3\frac{t_4^{2\beta}}{\Gamma(2\beta+1)}\right] \tag{75}$$

in $t_1 \geqslant t_2 \geqslant t_3 \geqslant t_4$. The prefactors arise from counting how often t_2, t_3 and t_4 arise as the smallest of the first two elements of any permutation of t_1,t_2,t_3,t_4. For the last summand, we additionally used the identity [[22] Eq. (15.1.20)]

$$F(\beta_1,-\beta_2;1+\beta_1;1)=\frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{\Gamma(\beta_1+\beta_2+1)}. \tag{76}$$

As a consistency check, we put $t_1=t_2$ and $t_3=t_4$ to get

$$\langle X^2(t_1)X^2(t_3)\rangle\simeq\frac{\mu_2^2}{\tau_0^{2\beta}}\left[5\frac{t_3^{2\beta}}{\Gamma(2\beta+1)}+\frac{t_3^\beta t_1^\beta}{\Gamma(\beta+1)^2}\times F\left(\beta,-\beta;1+\beta;\frac{t_3}{t_1}\right)\right]. \tag{77}$$

This can also be calculated by applying the introduced method to give

$$\frac{\partial^2}{\partial k_1^2}\frac{\partial^2}{\partial k_2^2}\breve{\omega}_2(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}}=\mu_2^2\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi(\Lambda_1)}{1-\phi(\Lambda_1)}+2\mu_2^2\left(\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}\right)^2+\frac{\mu_4}{2}\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}\simeq\frac{\mu_2^2}{\tau_0^{2\beta}}\left(\frac{1}{\Lambda_2^\beta}\frac{1}{\Lambda_1^\beta}+\frac{2}{\Lambda_2^{2\beta}}\right) \tag{78}$$

which subsequently reproduces Eq. (77) (by substituting t_3 for t_2). For Eq. (78) we used the generalized form of Eq. (59) shown in Appendix C [Eq. (C3)].

We want to close this subsection on the uncoupled case with the calculation of the long term behavior of $\langle X^2(t_1)X(t_2)\rangle$ as an example of an unsymmetric correlation. If $\mu_1=0$ this term vanishes. Therefore we consider a biased CTRW with $\mu_1\neq 0$. Since the correlation function is not symmetric with respect to index permutation, we have to work with $\breve{p}_2(\mathbf{k},\boldsymbol{\lambda})$ directly. We get in the long time limit

$$i\frac{\partial^2}{\partial k_1^2}\frac{\partial}{\partial k_2}\breve{p}_2(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}}\simeq\frac{\mu_1^3\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\left(\frac{\phi(\Lambda_{\{1\}})}{1-\phi(\Lambda_{\{1\}})}\right)^2+2\mu_1^3\left(\frac{\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\right)^2\frac{\phi(\Lambda_{\{1\}})}{1-\phi(\Lambda_{\{1\}})}+2\mu_1^3\left(\frac{\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\right)^2\frac{\phi(\Lambda_{\{2\}})}{1-\phi(\Lambda_{\{2\}})}$$
$$\simeq\frac{\mu_1^3}{\tau_0^3}\left(\frac{1}{\Lambda_{\{1,2\}}^\beta}\frac{1}{\Lambda_{\{1\}}^{2\beta}}+2\frac{1}{\Lambda_{\{1,2\}}^{2\beta}}\frac{1}{\Lambda_{\{1\}}^\beta}+2\frac{1}{\Lambda_{\{1,2\}}^{2\beta}}\frac{1}{\Lambda_{\{2\}}^\beta}\right). \tag{79}$$

The determination of Eq. (79) can be shortened by convincing oneself in advance that only the terms with $\phi_1(\cdots)$ are relevant for the long time behavior. The first two terms in the last line of Eq. (79) are the Laplace transforms of a function supported in $t_1 \geqslant t_2$ while the last term is the Laplace transform of a function supported in $t_2 \geqslant t_1$. The individual summands can be determined analogously to Eqs. (73) and (74). Putting these together with Eq. (76) gives the result, for $t_1 \geqslant t_2$,

$$\langle X^2(t_1)X(t_2)\rangle \simeq \frac{\mu_1^3}{\tau_0^3}\left[2\frac{t_2^{3\beta}}{\Gamma(3\beta+1)} + 2\frac{t_1^{\beta}}{\Gamma(\beta+1)}\frac{t_2^{2\beta}}{\Gamma(2\beta+1)}F\left(2\beta,-\beta;1+2\beta;\frac{t_2}{t_1}\right) + \frac{t_1^{2\beta}}{\Gamma(2\beta+1)}\frac{t_2^{\beta}}{\Gamma(\beta+1)}F\left(\beta,-2\beta;1+\beta;\frac{t_2}{t_1}\right)\right], \tag{80}$$

and for $t_2 \geqslant t_1$,

$$\langle X^2(t_1)X(t_2)\rangle \simeq \frac{\mu_1^3}{\tau_0^3}\left[3\frac{t_2^{3\beta}}{\Gamma(3\beta+1)} + 2\frac{t_1^{2\beta}}{\Gamma(2\beta+1)}\frac{t_2^{\beta}}{\Gamma(\beta+1)}F\left(2\beta,-\beta;1+2\beta;\frac{t_1}{t_2}\right)\right]. \tag{81}$$

B. The coupled case

In this subsection we allow a coupling between the step distribution and the waiting time. When we look at the n-point correlation and if the marginal distribution of the step size has all first n moments μ_q, we have

$$\lim_{\lambda\to 0}\phi_q(\lambda) = \mu_q. \tag{82}$$

Therefore, in the long time limit we end with the same terms as in the uncoupled case, except that we have to take the average (or marginal) moments.

Therefore we will concentrate here on the situation when not all marginal moments exist, but nevertheless for every bounded interval of waiting times, the step size distribution has all necessary moments (i.e., up to the nth moment if we look at the n-point correlation). To be able to calculate the long time limit, the $\phi_q(\lambda)$ have to exist for $\lambda > 0$, while they are allowed to diverge to $+\infty$ as $\lambda \to 0+$. One way to ensure this, is to put the condition that the increase of the moments with the waiting time is bounded by a polynomial.

Let us look at the Lévy walk [23]

$$\psi(x,t) = \phi(t)\frac{\delta(|x| - vt^{\kappa})}{2} \tag{83}$$

with $\phi(t)$ being for simplicity an one-sided Lévy stable distribution, which implies

$$\phi(t) \simeq \frac{\tau_0^{\beta}}{-\Gamma(-\beta)}\frac{1}{t^{1+\beta}} \quad \text{for } t\to\infty. \tag{84}$$

We should remark that we use another parameterization as in Ref. [23]. The long time behavior of the second and forth moment is

$$\phi_2(t) \simeq \frac{\tau_0^{\beta}v^2}{-\Gamma(-\beta)}\frac{t^{2\kappa}}{t^{1+\beta}},$$

$$\phi_4(t) \simeq \frac{\tau_0^{\beta}v^4}{-\Gamma(-\beta)}\frac{t^{4\kappa}}{t^{1+\beta}}. \tag{85}$$

We concentrate here on the case $\kappa > \frac{\beta}{2}$. Then the Tauberian theorems [20] give

$$\phi_2(\lambda) \simeq \frac{\tau_0^{\beta}v^2\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{2\kappa-\beta}},$$

$$\phi_4(\lambda) \simeq \frac{\tau_0^{\beta}v^4\Gamma(4\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{4\kappa-\beta}} \tag{86}$$

for $\lambda \to 0+$. Similar to Eq. (78) we get here

$$\begin{aligned}\frac{\partial^2}{\partial k_1^2}\frac{\partial^2}{\partial k_2^2}\breve{\omega}_2(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}} &= \frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_2(\Lambda_1)}{1-\phi(\Lambda_1)} + 2\left(\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)}\right)^2 + \frac{1}{2}\frac{\phi_4(\Lambda_2)}{1-\phi(\Lambda_2)} \\ &\simeq v^4\left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2\left(\frac{1}{\Lambda_2^{2\kappa}}\frac{1}{\Lambda_1^{2\kappa}} + \frac{2}{\Lambda_2^{4\kappa}}\right) + \frac{v^4}{2}\frac{\Gamma(4\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\Lambda_2^{4\kappa}}.\end{aligned} \tag{87}$$

In this case the term $\phi_4(\Lambda_2)/[1-\phi(\Lambda_2)]$ does not vanish in the long time limit. We get for $t_1 \geqslant t_2$

$$\langle X^2(t_1)X^2(t_2)\rangle \simeq D_1 v^4\frac{t_2^{4\kappa}}{\Gamma(4\kappa+1)} + D_2 v^4\frac{t_1^{2\kappa}t_2^{2\kappa}}{\Gamma(2\kappa+1)^2} \times F\left(2\kappa,-2\kappa;1+2\kappa;\frac{t_2}{t_1}\right), \tag{88}$$

with

$$D_1 = 5\left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2 + \frac{\Gamma(4\kappa-\beta)}{-\Gamma(-\beta)},$$

$$D_2 = \left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2. \tag{89}$$

Since the last example showed that in difference to the uncoupled case the term $\phi_4(\Lambda_2)/[1-\phi(\Lambda_2)]$ need not to be negligible for large times, one can ask, if it is possible that it dominates in this limit. For this we look at the following example [$\phi(t)$ as before]:

$$\psi(x,t) = \phi(t)\left(\frac{t^{2\kappa}}{1+t^{2\kappa}}\frac{\delta(|x|-v_1 t^{\kappa})}{2} + \frac{1}{1+t^{2\kappa}}\frac{\delta(|x|-v_2 t^{2\kappa})}{2}\right). \tag{90}$$

This corresponds to a CTRW where the walker makes after a waiting time t, either a step with modulus $v_1 t^{\kappa}$ or with $v_2 t^{2\kappa}$. Which of the step sizes is taken is chosen randomly with a probability of $t^{2\kappa}/(1+t^{2\kappa})$ for the first and a probability of $1/(1+t^{2\kappa})$ for the second. For $\kappa > \frac{\beta}{2}$ we get

$$\phi_2(\lambda) \simeq \tau_0^{\beta}(v_1^2+v_2^2)\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{2\kappa-\beta}},$$

$$\phi_4(\lambda) \simeq \tau_0^{\beta} v_2^4 \frac{\Gamma(6\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{6\kappa-\beta}}. \tag{91}$$

For the four-point correlation the term with $\phi_4(\cdots)$ dominates in the long time limit and we get [with $\phi_1(\lambda)=0$]

$$C_4(\mathbf{t}) \simeq v^4 \frac{\Gamma(6\kappa-\beta)}{-\Gamma(-\beta)}\frac{t_4^{6\kappa}}{\Gamma(6\kappa+1)} \tag{92}$$

for $t_1 \geqslant t_2 \geqslant t_3 \geqslant t_4$. The value is therefore determined by the earliest time only.

The scaling of the variance is $\langle X^2(t)\rangle \propto t^{2\kappa}$, therefore one would expect for the fourth moment a scaling $\propto t^{4\kappa}$ while it is $\langle X^4(t)\rangle \propto t^{6\kappa}$. This implies that the asymptotic one-point density

$$p_1^{\text{asymptotic}}(x) = \lim_{t\to\infty} t^{\kappa} p_1(t^{\kappa}x,t) \tag{93}$$

will have an infinity fourth moment. Nevertheless the fourth moment is well defined for all finite times t.

V. SUMMARY

We have introduced a diagrammatic method which allows to efficiently write down the joint probability distribution of a continuous-time random walk. While this method is more useful for calculations by hand, we additionally derived the recursion relation (31) for computer calculations. We exemplified the usability by calculating several multipoint probability distributions in Fourier-Laplace space. The analytic expressions for one and two point distributions given in the literature [11,14] were reproduced by means of our method. Additionally, we offered a direct proof of the fact [18] that the continuous-time random walk asymptotically becomes Markovian for finite mean waiting time in the unbiased case or for finite mean and variance of the waiting time in the biased case. Albeit we concentrate on the "leaper" model of continuous-time random walk, the adaptation to other models is straightforward. We show how to do the calculations for the "creeper" model in Appendix A.

In the second part, we focused on the determination of the multipoint correlation functions for the "leaper" model. Of course, it is always possible to derive them directly from the joint probability distributions, but this can be quite inefficient for many points. Therefore we derived a recursive procedure which allows the determination of the correlation functions without having to determine the joint probability distributions first. Finally, we showed the usage of this procedure on some standard examples of continuous-time random walks. The asymptotic expressions in the space-time uncoupled case can also be inferred from a different approach [15], they are also reproduced by our method.

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APPENDIX A: THE "CREEPER" MODEL

In the rest of this paper, we were mainly considering the "leaper" type for the continuous-time random walk. However, the methods introduced in Sec. II generalize directly to other models. In this appendix we want to show this on the example of the "creeper" model [19]. While for the "leaper" model the random walker stayed at the same x position during the waiting time and then leapt to its new position, the random walker in the "creeper" model moves during the waiting time with constant velocity to its new position.

The objective is now to find the equivalent expression for the contribution of one step. For this we start with the expression for $\eta_n^{(j)}(\mathbf{x},\mathbf{t})$ [the equivalent to Eq. (11)]. For the "creeper" model we have

$$\eta_n^{(j)}(\mathbf{x},\mathbf{t}) = \prod_{i:q_i=j}\delta\left(x_i - \frac{\chi_j}{\tau_j}t_i\right)\theta(\tau_j - t_i)\prod_{i:q_i<j}\delta(x_i)\delta(t_i) \times \prod_{i:q_i>j}\delta(x_i-\chi_j)\delta(t_i-\tau_j). \tag{A1}$$

We want to motivate this expression: the change of model does only affect the behavior during the waiting time, but not the step sizes and waiting time of a completed step. Therefore we can expect, that we have a change in $\eta_n^{(j)}(\mathbf{x},\mathbf{t})$ only in the indices i with $q_i=j$ (that are the times which are in the waiting time of the jth step—the step we are looking at). The difference for these is that they do not stay at the same position, but move with constant velocity $\frac{\chi_j}{\tau_j}$, where χ_j and τ_j are the step size and the waiting time, respectively, for the current step. This motivates why we have to replace $\delta(x_i)$ with $\delta(x_i-\frac{\chi_j}{\tau_j}t_i)$.

We can now use the same machine as in Sec. II. The contribution of a horizontal line does not change, since they consist of steps with no indices i fulfilling $q_i=j$. The contribution of a vertex becomes

$$\rho_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) = \int d\chi d\tau \psi(\chi,\tau) e^{-\Lambda_{\mathcal{L}}\tau + iK_{\mathcal{L}}\chi}\prod_{v\in\mathcal{V}}\frac{1-e^{-\lambda_v\tau+ik_v\chi}}{\lambda_v - ik_v\frac{\chi}{\tau}}. \tag{A2}$$

The one point probability distribution can be again calculated from the simple diagram having only one horizontal line and one vertex

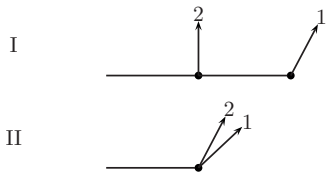

FIG. 4. The diagrams relevant for the calculation of Eq. (A4).

$$p_1(k,\lambda) = \frac{1}{1-\psi(k,\lambda)} \int d\chi d\tau \psi(\chi,\tau) \frac{1-e^{-\lambda\tau+ik\chi}}{\lambda - ik\frac{\chi}{\tau}}. \quad \text{(A3)}$$

This expression coincides with the one given by Hughes in Ref. [19] (p. 287f), where he also shows how to obtain the asymptotic behavior.

In equivalence to Eq. (25) we can also define the contributions to $\omega_n(\mathbf{k},\boldsymbol{\lambda})$. With the diagrams given in Fig. 4 we get

$$\omega_2^{\mathrm{I}}(\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)} \times \int d\chi d\tau \psi(\chi,\tau) \frac{(1-e^{-\lambda_2\tau+ik_2\chi})e^{-\lambda_1\tau+ik_1\chi}}{\lambda_2 - ik_2\frac{\chi}{\tau}} \times \frac{1}{1-\psi(k_1,\lambda_1)} \int d\chi d\tau \psi(\chi,\tau) \frac{1-e^{-\lambda_1\tau+ik_1\chi}}{\lambda_1 - ik_1\frac{\chi}{\tau}},$$

$$\omega_2^{\mathrm{II}}(\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{2}\frac{1}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)} \times \int d\chi d\tau \psi(\chi,\tau) \frac{1-e^{-\lambda_1\tau+ik_1\chi}}{\lambda_1 - ik_1\frac{\chi}{\tau}} \frac{1-e^{-\lambda_2\tau+ik_2\chi}}{\lambda_2 - ik_2\frac{\chi}{\tau}},$$

$$\omega_2(\mathbf{k},\boldsymbol{\lambda}) = \omega_2^{\mathrm{I}}(\mathbf{k},\boldsymbol{\lambda}) + \omega_2^{\mathrm{II}}(\mathbf{k},\boldsymbol{\lambda}). \quad \text{(A4)}$$

The two point probability distribution is then given by symmetrization

$$p_2(k_1,k_2,\lambda_1,\lambda_2) = \omega_2(k_1,k_2,\lambda_1,\lambda_2) + \omega_2(k_2,k_1,\lambda_2,\lambda_1). \quad \text{(A5)}$$

APPENDIX B: RENEWAL EQUATION APPROACH

In this appendix, we want to show an alternative derivation for the joint probability distributions of a continuous-time random walk. We concentrate on the probability corresponding to a given diagram and use a renewal equation approach to get a recursion relation on the number of vertices. Figure 5 shows the relevant diagrams for this calculation. The sets $\mathcal{V}_j$ describe the indices leaving the jth vertex (counting from the right). The diagram D_r in the first line contains all r vertices while the diagram D_{r-1} in the second line follows from this by removing the leftmost vertex. The

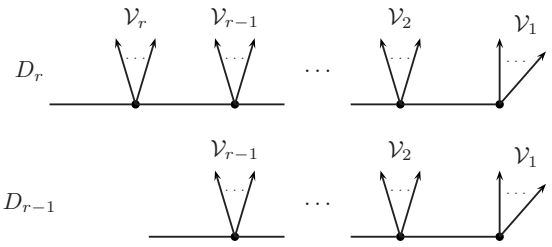

FIG. 5. The two diagrams which appear in the renewal equation in Appendix B. The sets $\mathcal{V}_j$ contain the indices of the jth vertex (from the right). The second diagram stems from the first one by removing the leftmost vertex.

corresponding joint probability distributions are denoted $p_{D_r}(\mathbf{x},\mathbf{t})$ and $p_{D_{r-1}}(\mathbf{x},\mathbf{t})$, respectively. The renewal equation is set up by splitting the process in the first step (the first renewal) and the rest. This step can either contain all times t_v ($v \in \mathcal{V}_r$) and then continue with the shifted probability for the diagram D_{r-1}, or all times t_v ($v \in \mathcal{V}_r$) are in a later step which means that after the first step the situation is still described by the diagram D_r. The renewal equation then reads

$$p_{D_r}(\mathbf{x},\mathbf{t}) = \int d\chi d\tau \psi(\chi,\tau) \prod_{v\in\mathcal{V}_r} [\theta(\tau - t_v)\delta(x_v)] \times p_{D_{r-1}}(\mathbf{x}-\chi,\mathbf{t}-\tau) + \int d\chi d\tau \psi(\chi,\tau) p_{D_r}(\mathbf{x}-\chi,\mathbf{t}-\tau), \quad \text{(B1)}$$

where we write $\mathbf{x}-\chi$ and $\mathbf{t}-\tau$ for $(x_1-\chi,x_2-\chi,\ldots)$ and $(t_1-\tau,t_2-\tau,\ldots)$, respectively. Taking the Fourier-Laplace transform of Eq. (B1) and solving for $p_{D_r}(\mathbf{k},\boldsymbol{\lambda})$ gives (with $\mathcal{L}_r = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{r-1}$)

$$p_{D_r}(\mathbf{k},\boldsymbol{\lambda}) = \frac{\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V}_r)} (-1)^{|\mathcal{J}|} \psi(K_{\mathcal{L}_r}, \Lambda_{\mathcal{L}_r} + \Lambda_{\mathcal{J}})}{[1-\psi(K_{\mathcal{V}_r\cup\mathcal{L}_r})]\prod_{v\in\mathcal{V}_r}\lambda_v} p_{D_{r-1}}(\mathbf{k},\boldsymbol{\lambda}). \quad \text{(B2)}$$

The factor in front of $p_{D_{r-1}}(\mathbf{k},\boldsymbol{\lambda})$ is identical to the contribution of the leftmost line and vertex as calculated in Sec. II. Iterating Eq. (B2) therefore reproduces the result from Sec. II for $p_{D_r}(\mathbf{k},\boldsymbol{\lambda})$.

APPENDIX C: DERIVATION OF EQ. (59)

In this appendix we derive a more general version of Eq. (59). For this we introduce the operator $\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})]$ for a function $f(\mathbf{k},\boldsymbol{\lambda})$ as

$$\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})] = \left(\frac{\partial}{i\partial k_1}\right)^{q_1} \cdots \left(\frac{\partial}{i\partial k_m}\right)^{q_m} f(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}}. \quad \text{(C1)}$$

With this notation, we can write $\mathcal{D}^m[p_n(\mathbf{k},\boldsymbol{\lambda})]$ for the Laplace transform of $\langle X^{q_1}(t_1)\cdots X^{q_m}(t_m)\rangle$ (we suppress the $\mathbf{q}$ dependence for simplicity). We additionally define a permuted version of $\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})]$ for any $\sigma \in S_n$ (S_n is the symmetric group of n elements):

$$\mathcal{D}^m_\sigma[f(\mathbf{k},\boldsymbol{\lambda})] = \left(\frac{\partial}{i\partial k_{\sigma(1)}}\right)^{q_1}\cdots\left(\frac{\partial}{i\partial k_{\sigma(m)}}\right)^{q_m} f(\mathbf{k},\boldsymbol{\lambda})\Bigg|_{\mathbf{k}=\mathbf{0}}. \tag{C2}$$

When calculating correlation functions with the method introduced in Sec. IV one encounters terms such as $\mathcal{D}^m[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})]$ with $m \leqslant n$. For $m<n$ the problem arises that for the recursive procedure of Sec. IV one would need a differentiation of $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ with respect to k_n. In the case of symmetric differential operators it is possible to circumvent the problem by use of the equation

$$\sum_{\sigma\in S_n} \mathcal{D}^m_\sigma[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})] = \sum_{\theta\in S_m} \mathcal{D}^m_\theta[\breve{\omega}_m(\mathbf{k},\boldsymbol{\lambda})], \tag{C3}$$

where we identify S_m as a subgroup of S_n in the standard way by acting on the first m elements. In case of unsymmetric differential operators, it is better to work with $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ instead of $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ since then the reduction is simply Eq. (C8) [in fact, the derivation of Eq. (C3) is done by tracing back the problem to Eq. (C8)]. Equation (59) follows then from Eq. (C3) by the following identities (here $q_1=\ldots=q_{n-d}=1$):

$$\sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\})\\ |\mathcal{J}|=n-d}} \frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}}\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\Bigg|_{\mathbf{k}=\mathbf{0}} = \frac{1}{d!(n-d)!}\sum_{\sigma\in S_n}\mathcal{D}^{n-d}_\sigma[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})], \tag{C4}$$

and

$$\breve{\gamma}_{n-d}(\boldsymbol{\lambda}) = \frac{1}{(n-d)!}\sum_{\theta\in S_{n-d}}\mathcal{D}^{n-d}_\theta[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})]. \tag{C5}$$

For the derivation of Eq. (C3), we use the notation $\pi\mathbf{k}$ with $\pi\in S_n$ for the permutation of the different elements of the vector (i.e., π acts on the indices). This allows to write Eq. (27) as

$$\breve{p}_n(\mathbf{k},\boldsymbol{\lambda}) = \sum_{\pi\in S_n}\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda}). \tag{C6}$$

Symmetrizing Eq. (C3) by adding the left hand side over all permutations $\pi\in S_n$ of the arguments, we get

$$\begin{aligned}\sum_{\pi\in S_n}\sum_{\sigma\in S_n}\mathcal{D}^m_{\pi\sigma}[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})] &= \sum_{\pi,\eta\in S_n}\mathcal{D}^m_\eta[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\mathcal{D}^m_\eta[\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\mathcal{D}^m_\eta[\breve{p}_n(\eta\mathbf{k},\eta\boldsymbol{\lambda})].\end{aligned} \tag{C7}$$

Here we have used the fact that $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ is symmetric in the arguments. The term $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ can be simplified by noting that the integrating of one parameter in a joint probability distribution gives a joint probability distribution with lesser points, i.e.,

$$\begin{aligned}&\breve{p}_n(k_1,\ldots,k_{n-1},k_n=0,\lambda_1,\ldots,\lambda_n)\\ &\quad=\breve{p}_{n-1}(k_1,\ldots,k_{n-1},\lambda_1,\ldots,\lambda_{n-1})\end{aligned} \tag{C8}$$

and similar relations for other $k_i=0$. Continuing Eq. (C7) gives

$$\begin{aligned}\sum_{\pi\in S_n}\sum_{\sigma\in S_n}\mathcal{D}^m_{\pi\sigma}[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})] &= \sum_{\eta\in S_n}\mathcal{D}^m_\eta[\breve{p}_m(\pi'\mathbf{k},\eta\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\sum_{\theta\in S_m}\mathcal{D}^m_\eta[\breve{\omega}_m(\eta\theta^{-1}\mathbf{k},\eta\theta^{-1}\boldsymbol{\lambda})]\\ &= \sum_{\pi\in S_n}\sum_{\theta\in S_m}\mathcal{D}^m_{\pi\theta}[\breve{\omega}_m(\pi\mathbf{k},\pi\boldsymbol{\lambda})].\end{aligned} \tag{C9}$$

Equation (C3) follows now from Eq. (C9) by noting that the right-hand side of Eq. (C3) has support in the domain $t_1 \geqslant\cdots\geqslant t_n$ and the only term in the sum in the last line of Eq. (C9) which shares this property is $\breve{\gamma}_l(\boldsymbol{\lambda})$ (one can assure oneself that this argument is not invalidated by the fact that these domains overlap when some t_i are equal since then the values are still fixed for symmetry reasons).

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[24] Being more precise, the two Heaviside θ functions in Eq. (6) should be replaced with a function which is one for $\tau_1+\cdots+\tau_{q_i-1}\leqslant t_i<\tau_1+\cdots+\tau_{q_i}$ and zero otherwise. This would only change countable many points and therefore we will not see any difference in the Fourier-Laplace transform.